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# Poverty Effects of the Minimum Wage: The Role of Household Employment Composition

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# Poverty Effects of the Minimum Wage: The Role of Household Employment Composition

## **Abstract**

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## **Keywords**

poverty, employment, unemployment, minimum wage, household composition

## **Comments**

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## **Poverty Effects of the Minimum Wage: The Role of Household Employment Composition \***

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**November, 2007**

### **Abstract**

A change in a country's minimum wage will in general affect the number of workers in covered sector employment, uncovered sector employment, and unemployment. The impact of these labor market adjustments on absolute poverty will depend on how the pattern of employment composition changes within households and on how income is shared within households. An earlier paper (Fields and Kanbur, 2007) focused on the income-sharing dimension of the problem. The present paper focuses on household employment composition. For a particular structure of the labor market— one with good jobs, bad jobs, unemployment, and adult and youth workers— and with a particular model of how the sectoral patterns of employment are translated into household employment composition, we analyze the impact of minimum wages on a class of absolute poverty measures. The precise characterizations demonstrate the need for a nuanced appreciation of the impacts of a minimum wage increase, since they depend intricately on the values of key parameters (the poverty line, poverty aversion, labor demand elasticity, and the starting level of the minimum wage). Moreover, the relationship between poverty and the minimum wage is in general non-monotonic, so that local effects can be quite different from the effects of large changes in the minimum wage.

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## Poverty Effects of the Minimum Wage: The Role of Household Employment Composition

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### I. Introduction

Minimum wages are commonly evaluated by labor economists in one of two ways. Some analysts pay primary attention to the fact that a higher minimum wage increases the labor market earnings of those employed, while others emphasize that a higher minimum wage would normally be expected to reduce the number employed (Brown, 1999; Ehrenberg and Smith, 2006; Borjas, 2005). However, an analysis of the effects of these labor market consequences on poverty, which is the ultimate focus of much of the policy discourse, requires two further steps. First, the employment composition of the labor market has to be translated into the employment composition of each household. Second, a method of income sharing within the household must be specified.

In a previous paper (Fields and Kanbur, 2007), in a model with only two types of workers - employed and unemployed - we focused primarily on different ways that incomes might be shared within households and how each affected the impact of minimum wages on poverty. In the present paper we assume perfectly equal income sharing within the household, and focus instead on employment composition. We develop the household distribution of income from the labor market outcomes for a model with good jobs, bad jobs and unemployment, and adults and youths searching for jobs. Such a structure allows us, for example, to incorporate the fact that in countries such as the United States, many minimum wage workers live in non-poor households (Burkhauser, Couch, and Wittenburg, 2000). The impact of a minimum wage on poverty then depends crucially on the employment composition of households at different levels of income. We ask, when exactly does a higher minimum wage *raise* poverty, when does it *lower* poverty, and when is poverty *unchanged*?

The remainder of the paper is structured as follows. Section II presents the main features of the model. Section III derives the effect of a small increase in the minimum wage. Section IV extends the analysis to large changes in the minimum wage. Section V summarizes and concludes.

## **II. The Model**

### **A. The Labor Market and Household Employment Composition**

In this paper, it is assumed that there is a fixed number of households, normalized at 1. Each household consists of two household members: one adult and one youth. Thus, the total labor supply is 2.

The labor market has two types of jobs. High wage jobs,  $h$ , pay a wage  $w_h$ . The wage of these “good jobs” is assumed to be invariant to any changes taking place elsewhere in the labor market. Employment in the high wage sector, denoted  $x_h$ , is determined according to a standard downward-sloping labor demand curve  $x_h = f(\hat{w}_h)$ ,  $f' < 0$ . Low wage jobs,  $l$ , pay a minimum wage  $\hat{w}_l$ , which is determined as a matter of public policy. Employment in these “bad jobs” in the low wage sector is also determined according to a standard downward-sloping labor demand curve  $x_l = g(\hat{w}_l)$ ,  $g' < 0$ . It is assumed that only adults can be employed in the high wage sector. Adults who fail to find employment in the high wage sector, together with youths, form an undifferentiated pool of applicants for low wage jobs.

The low wage  $\hat{w}_l$  is of course less than the high wage  $w_h$ , and households in which both members are employed earn more than households in which only one is employed. In addition, we assume that the low wage is greater than half the high wage. Together, these assumptions imply that

$$0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

These inequalities will be maintained throughout this paper.

We now discuss the number of persons earning each of these amounts and the per capita household incomes. Employment in the high wage and low wage sectors are respectively  $x_h$  and  $x_l$ . Given that the high wage sector employs only adults, the number of whom is normalized at 1,

the number of adults seeking low wage jobs is  $(1 - x_h)$ . In addition, all youth (the number of which is normalized at 1) also seek low wage jobs. Thus, the number of applicants for low wage jobs is  $2 - x_h$ , and the probability that a low wage applicant gets a job is  $\frac{x_l}{2 - x_h}$ . An adult can be employed in a high wage job with probability  $x_h$ , employed in a low wage job with probability  $(1 - x_h)(\frac{x_l}{2 - x_h})$ , or unemployed with probability  $(1 - x_h)(1 - \frac{x_l}{2 - x_h})$ . A youth can be employed in a low wage job with probability  $\frac{x_l}{2 - x_h}$  or unemployed with probability  $(1 - \frac{x_l}{2 - x_h})$ . Putting these respective wages and employment probabilities together, we have six possible types of households, where  $A_i$ ,  $i = h, l, u$  is the employment state of the adult and  $Y_j$ ,  $j = l, u$  is the employment state of the youth; see Table 1. All household members are assumed to share their earnings. Hence household earnings per capita is the relevant measure of the well-being of each individual in the household. Clearly the poorest individuals are those who live in households where nobody works (H6). Next come individuals in households where one member is unemployed but the other member is employed in the minimum wage sector (H4 and H5). Given our assumption that the high wage is less than twice the low wage, the case where the adult has a high wage job but the youth is unemployed (H3) gives lower per capita income than the case where both the adult and the youth are employed in the low wage sector (H2). Finally, the highest household per capita income occurs when the adult has a good job and the youth is employed in the minimum wage sector (H1). Table 1 sets out, therefore, the income distribution in this society. We turn now to the measurement of poverty based on this income distribution.

### **B. How Poverty Is Measured**

Poverty in this paper is measured in absolute terms. The analysis consists of determining how poverty in the labor market varies with changes in  $\hat{w}_l$ . Poverty is gauged by comparing the household's labor market earnings to a fixed poverty line  $z$ . The poverty line is  $\$z$  per person, i.e.,  $\$2z$  per household.

How high the fixed poverty line is itself allowed to vary. Five cases are analyzed in this paper. Moving from the lowest poverty line to the highest, they are:

$$\text{Case 1: } 0 < z < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 2: } 0 < \frac{\hat{w}_l}{2} < z < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 3: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < z < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 4: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < z < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 5: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2} < z.$$

Case 1 is where the poverty line is so low that only households with all members unemployed are poor. Case 2 brings into the poverty net those households where one member is unemployed but the other member has a minimum wage job. These households will benefit from a rise in the minimum wage if they hold onto the minimum wage job. Case 3 widens the poverty net still further to include households where the adult is employed in the high wage sector but the youth is unemployed. Case 4 sets the poverty line at a sufficiently high level that income from two minimum wage jobs is not enough to pull the household out of poverty. Finally, Case 5 is the extreme case where the poverty line is so high that everybody is in poverty. Observers who argue that the minimum wage does not target poverty very well are clearly thinking of Cases 1 through 4, in which non-poor households have minimum wage earners. But in Cases 2 through 5, poor households also have minimum wage workers. Hence in Cases 2, 3 and 4, minimum wage workers are to be found in both poor and non-poor households.

In all cases, poverty is gauged using the class of absolute poverty indices developed by Foster, Greer, and Thorbecke (1984). The FGT index, denoted  $P_\alpha$ , takes each poor person's poverty deficit as a percentage of the poverty line, raises it to a power  $\alpha$ , and averages over the entire population. Letting  $y_i$  be the income of the  $i$ -th person,  $z$  the poverty line,  $q$  the number of poor persons, and  $n$  the total number of persons, the  $P_\alpha$  poverty measure is given by:

$$P_\alpha = \frac{1}{n} \sum_{i=1}^q \left( \frac{z - y_i}{z} \right)^\alpha. \quad (1)$$

Three specific values of  $\alpha$  are of particular interest. As is well known, when  $\alpha = 0$  this measure collapses to the headcount ratio, the fraction of people below the poverty line. Other interesting values of  $\alpha$  are when  $\alpha$  is greater than or equal to one. Benchmark values in this range are  $\alpha = 1$ , in which case we have the income gap measure of poverty, and  $\alpha = 2$ , which is known as the squared poverty gap measure. The higher is  $\alpha$ , the greater is the sensitivity of poverty to changes in the incomes of the poorest compared to the incomes of the not so poor. For these reasons,  $\alpha$  is known as the poverty aversion parameter. To allow for the social loss from poverty to increase at an increasing rate as incomes fall relative to the poverty line,  $\alpha$  must be greater than 1. Because of the intuitive appeal of integer values of  $\alpha$ , it is common for empirical poverty researchers to choose  $\alpha = 2$ . Different degrees of poverty aversion will be seen to be important in delineating the consequences of the minimum wage for poverty.

We turn now to the poverty effects of higher minimum wages in this model.

### **III. The Poverty Effects of a Higher Minimum Wage Within Each of the Five Cases**

We have set forth five cases above. For each of these five cases, different types of tradeoffs are involved in raising the minimum wage. The results are summarized in Table 2. The detailed derivations are given in the Appendix 1. Here we will provide an intuitive discussion of the results. The results fall into three groups and will be discussed accordingly: 1) The results for  $\alpha = 0$ , in which  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ . 2) The results for Case 1, also in which  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ . 3) The results for  $\alpha \geq 1$  in Cases 2 through 5, in which  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  ( $< 0$ ) if the elasticity of labor demand in the minimum wage sector  $\eta$  is sufficiently high (low).

The first set of results (for  $\alpha = 0$ ) can be understood in a similar way for all five cases. When  $\alpha = 0$ , the poverty measure being used is the poverty headcount ratio. A higher minimum wage causes more people to become unemployed, which raises the number of households in



poverty, i.e.,  $\frac{dP_0}{d\hat{w}_l} > 0$ . Given that the  $P_0$  poverty measure focuses only on the numbers in poverty and not on how poor the poor are, the gains to the incomes of poor working households is not counted, and poverty (measured by the number in poverty) always rises. The only reason that  $\frac{dP_0}{d\hat{w}_l} = 0$  (in Case 5) is that the poverty line is so high that everybody is in poverty to begin with, and so no further increase in poverty is possible.

The second set of results is for Case 1, i.e., the case in which the only poor households are those for which both household members are unemployed. Thus an increase in the minimum wage cannot possibly affect their incomes, but their numbers will increase with the rise in unemployment. Thus, no matter what the value of  $\alpha$ , in this case, an increase in the minimum wage will increase poverty, i.e.,  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ .

The third set of results is for  $\alpha \geq 1$  in Cases 2 through 5. In each of these cells,  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  when  $\eta$  is sufficiently high and  $\frac{dP_\alpha}{d\hat{w}_l} < 0$  when  $\eta$  is sufficiently low. That is, when the elasticity of labor demand is greater than the critical value corresponding to that particular case, as the minimum wage increases, poverty will increase. Poverty will rise when the unemployment effect of a minimum wage increase dominates the earnings effect. Of course, this is more likely the greater the elasticity of demand for labor. On the other hand, when the elasticity of labor demand is less than the critical value, as the minimum wage increases, poverty will decrease: the earnings effect dominates the unemployment effect.

This completes our analysis of how poverty changes locally with the minimum wage *within* each of the five cases. Let us now analyze what happens when changes in the minimum wage are so large that we move across cases.

#### **IV. The Poverty Effects of a Large Increase in the Minimum Wage**

Section III analyzed the effects of an infinitesimal increase in the minimum wage. In this section, we ask what happens if the minimum is increased discretely. On the one hand, the

discrete jump in the minimum wage can occur within a case. When this happens, the effect of the minimum wage on poverty is the integral of all the infinitesimal changes. No new analysis is needed when this happens. On the other hand, the discrete jump in the minimum wage can cause the economy to switch from one case to another. We show in this section that when such a switch occurs, the change in poverty may be discontinuous and, moreover, may go in the opposite direction from what happens on either side of the discontinuity.

### **A. Two Examples**

It is possible to gain further insights by looking at specific numerical examples. These examples will then be used to derive more general results.

The two examples we present are similar in most respects. They have the same high wage  $\hat{w}_h = 15$ , the same employment at the high wage  $x_h = 0.1$ , the same range of possible minimum wages (from  $\frac{\hat{w}_h}{2} = 7.5$  to  $\hat{w}_h = 15$ ), the same constant elasticity of demand for labor in the low wage sector  $\eta = 0.7$ , and the same demand for labor curve in the low wage sector  $x_l = 0.3 - 0.7 \ln \hat{w}_l$ . The two examples differ in one important respect, however: in Example 1, the poverty line  $z$  is in the range  $z < \frac{w_h}{2}$ , while in Example 2, the poverty line  $z$  is in the range  $z > \frac{w_h}{2}$ . (Note: In Cases 1 and 2,  $z < \frac{w_h}{2}$ , while in Cases 3 through 5,  $z > \frac{w_h}{2}$ .) For the calculations below,  $z = 5$  in Example 1 and 12.5 in Example 2.

To analyze how poverty as measured by  $P_\alpha$  changes with  $\frac{\hat{w}_l}{z}$ , our strategy is to fix  $z$  and raise  $\hat{w}_l$  from the lowest possible value to the highest possible value. We do this first when  $z < \frac{w_h}{2}$  and then when  $z > \frac{w_h}{2}$ .

### **B. Analysis for the Poverty Headcount Ratio ( $\alpha = 0$ )**

We start with the situation where  $\alpha$  is chosen to equal 0, i.e., the poverty measure is the headcount ratio. The headcount ratio is sensitive only to the number of people below the poverty

line but not to the severity of their poverty. This means that changing the minimum wage induces only an unemployment effect but no earnings effect.

When  $P_\alpha = 0$ , the unemployment effect operates in the same way in Cases 1 through 4: an increase in the minimum wage reduces employment in the low wage sector, thereby increasing poverty as long as we remain within any of these four cases. In Case 5, however, everyone is poor and remains so, and therefore a change in the minimum wage has no effect on the poverty headcount.

What happens within a case is not the same as what happens in moving from one case to the next. To illustrate this point, consider Figures 1 and 2.

Figure 1 graphs the poverty headcount ratio  $P_\theta$  in Example 1. We see that  $P_\theta$  increases as the minimum wage rises within Case 2. However, there is a discontinuous fall in  $P_\theta$  at  $\hat{w}_l = 10$ . Why 10? Because that is twice the poverty line (5 in Example 1), which is the boundary between Case 2 and Case 1. When the minimum wage rises above 10, all of the people living in households with just one member employed at the minimum wage suddenly escape from poverty. We are now in the range of Case 1. In that range, a further increase of the minimum wage decreases employment and therefore raises the poverty headcount. This range ends just before the minimum wage equals the high wage, i.e., as  $\hat{w}_l \rightarrow w_h$ .

Suppose we continue to maintain that  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}$  but now  $z > \frac{w_h}{2}$ .

These conditions hold in Example 2. Figure 2 graphs the poverty headcount ratio  $P_\theta$  in Example 2. The figure shows that as the minimum wage rises,  $P_\theta$  is constant (at 1) in Case 5 and increases within Cases 4 and 3. It also shows discontinuous drops at the boundaries of the Cases. The reason is analogous to Example 1. At the boundary between Cases 5 and 4, all of the households with the maximum possible earnings – that is, those in which the adult is employed in a high wage job and the youth in a low wage job – suddenly escape poverty. Similarly, at the boundary between Cases 4 and 3, those households in which both the adult and the youth are employed in low wage jobs suddenly escape poverty.

These examples illustrate results that are quite general:

*Proposition 1: When  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}$  and  $z < \frac{w_h}{2}$ , an increase*

*in the minimum wage raises  $P_0$  within a case but may lower  $P_0$  if the economy crosses from Case 2 to Case 1.*

Proof: In Appendix 2

Turning now to the case exemplified by Figure 2, we have the following general result:

*Proposition 2: When  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}$  and  $z > \frac{w_h}{2}$ , an increase*

*in the minimum wage leaves  $P_0$  unchanged if the minimum wage remains within Case 5, raises  $P_0$  if the minimum wage remains within Case 4 or Case 3, and may lower  $P_0$  if the economy crosses from Case 5 to Case 4 or from Case 4 to Case 3.*

Proof: In Appendix 2

This completes our analysis of how the poverty headcount ratio  $P_0$  varies with the minimum wage  $\hat{w}_l$ . We turn now to the analysis of the situation where poverty is measured by the squared poverty gap  $P_2$ .

### **C. Analysis for the Squared Poverty Gap ( $\alpha = 2$ )**

The squared poverty gap  $P_2$  is sensitive both to the number of people below the poverty line and to the severity of their poverty. Changing the minimum wage will induce both an unemployment effect and an earnings effect. As detailed in Section III, poverty as measured by  $P_2$  may increase or decrease depending on the relative size of these two effects.

Figure 3 graphs the squared poverty gap  $P_2$  in Example 1. In this particular example, as the minimum wage increases,  $P_2$  increases in both Cases 2 and 1. This is not a general result:  $P_2$

could be increasing, decreasing, or change sign within either of the two Cases. Figure 4 graphs the squared poverty gap  $P_2$  in Example 2. In this particular example, we have a U-shaped pattern: as the minimum wage increases,  $P_2$  decreases in Case 5, decreases and then increases in Case 4, and increases throughout Case 3. This U shape is not a general result:  $P_2$  could be decreasing throughout, increasing throughout, or change sign depending on parameter values. The general result is:

*Proposition 3: When  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}$ ,  $P_2$  is a continuous function of the minimum wage  $\hat{w}_l$ .*

Proof: In Appendix 2

Although the behavior of  $P_2$  with respect to the minimum wage is continuous, it can be non-monotonic, as shown in Figure 4. This once again means that local findings, whether theoretical or empirical, are not necessarily a good guide to the implications of discrete changes. Thus, in Figure 4, while a small increase in the minimum wage for low values of the wage may lower poverty, a sufficiently large increase may have the opposite effect. On the other hand, just because an increase in the minimum wage from a particular starting point is observed to increase poverty is no guarantee that an increase in the minimum wage will have the same effect as an increase in the minimum wage from some other starting point.

## **V. Conclusion**

Fields and Kanbur (2007) brought the issue of income-sharing within the household to the forefront of the debate on the poverty impact of minimum wages. That paper showed how this poverty impact depends crucially on the income-sharing rule.

In this paper, the following model has been used. We have assumed equal sharing within the household to highlight the importance of the household employment composition. Each household consists of one adult and one youth. There are two types of jobs, high wage jobs and low wage jobs. The minimum wage applies to low wage jobs. Only adults may be hired for the high wage jobs. Those adults not hired for the high wage jobs and all youth compete for the low wage jobs. Of these, the ones not hired in the low wage jobs are unemployed. This structure determines the employment composition of each household, which in turn determines its income. A household is poor if and only if its per capita earnings are below a pre-established poverty line.

We showed that a minimum wage increase can raise poverty, lower poverty, or leave poverty unchanged. The particular outcome depends on the specific balance between the high wage, the low wage, employment in high-wage and low-wage jobs, the elasticity of demand for labor with respect to the minimum wage, and the value of  $\alpha$  chosen.

Table 2 summarizes the patterns that arise depending on how high the poverty line is and which value of  $\alpha$  is chosen. The fifteen cells of Table 2 reflect what happens *within* a case. In addition, minimum wage changes may be large enough to cause *movements across cases*. We proved three propositions relating to movements across cases, showing that  $P_0$  necessarily changes *discontinuously* when crossing cases and that  $P_2$  necessarily changes *continuously* when crossing cases. Furthermore, we demonstrated that there may be non-monotonicities in the relationship, which means that local results—theoretical or empirical—are not necessarily a good guide to the effects of discrete changes.

The results derived here reinforce the general conclusion from Fields and Kanbur (2007) that no simple statement can be made about whether an increase in the minimum wage raises poverty, lowers poverty, or leaves poverty unchanged. A detailed analysis is needed before conclusions can be drawn. This strongly suggests that the nature of the policy debate should shift from the simplistic “yes” versus “no” format that is current to a more nuanced discussion of the precise conditions under which a minimum wage will or will not reduce poverty.

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**Table 1.**  
**Types of Households and Distribution of Earnings.**

Type of household	Number of occurrences	Total household earnings	Household earnings per capita
H1. $(A_h, Y_l)$	$x_h \frac{x_l}{2 - x_h}$	$w_h + \hat{w}_l$	$\frac{w_h + \hat{w}_l}{2}$
H2. $(A_l, Y_l)$	$(1 - x_h) \left( \frac{x_l}{2 - x_h} \right) \left( \frac{x_l}{2 - x_h} \right)$	$2\hat{w}_l$	$\hat{w}_l$
H3. $(A_h, Y_u)$	$x_h \left( 1 - \frac{x_l}{2 - x_h} \right)$	$w_h$	$\frac{w_h}{2}$
H4. $(A_l, Y_u)$	$(1 - x_h) \left( \frac{x_l}{2 - x_h} \right) \left( 1 - \frac{x_l}{2 - x_h} \right)$	$\hat{w}_l$	$\frac{\hat{w}_l}{2}$
H5. $(A_u, Y_l)$	$(1 - x_h) \left( 1 - \frac{x_l}{2 - x_h} \right) \left( \frac{x_l}{2 - x_h} \right)$	$\hat{w}_l$	$\frac{\hat{w}_l}{2}$
H6. $(A_u, Y_u)$	$(1 - x_h) \left( 1 - \frac{x_l}{2 - x_h} \right) \left( 1 - \frac{x_l}{2 - x_h} \right)$	0	0



**Table 2.**  
**Summary of Results Concerning the Effect of a Minimum Wage Increase on Poverty as Gauged by  $P_\alpha$ .**

	Case 1	Case 2	Case 3	Case 4	Case 5
$\alpha = 0$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	$\frac{dP_\alpha}{d\hat{w}_l} = 0$
$\alpha = 1$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta \geq 1$ (<1), $\frac{dP_\alpha}{d\hat{w}_l} \geq 0$ (<0).
$\alpha > 1$	$\frac{dP_\alpha}{d\hat{w}_l} > 0$	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).	When $\eta$ is sufficiently high (low), $\frac{dP_\alpha}{d\hat{w}_l} > 0$ (<0).

Note: The parameter  $\eta$  is the wage elasticity of labor demand in the minimum wage sector. Moving from the lowest poverty line to the highest, the five cases are:

$$\text{Case 1: } 0 < z < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 2: } 0 < \frac{\hat{w}_l}{2} < z < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 3: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < z < \hat{w}_l < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 4: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < z < \frac{\hat{w}_l + w_h}{2}.$$

$$\text{Case 5: } 0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{\hat{w}_l + w_h}{2} < z.$$

Figure 1.

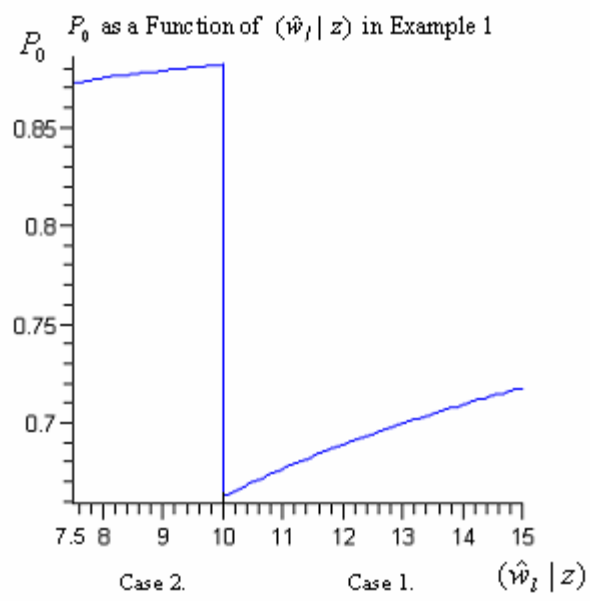


Figure 2.

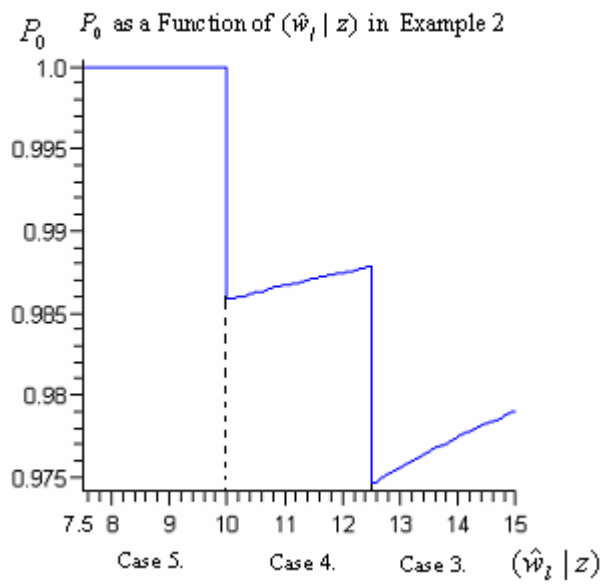


Figure 3.

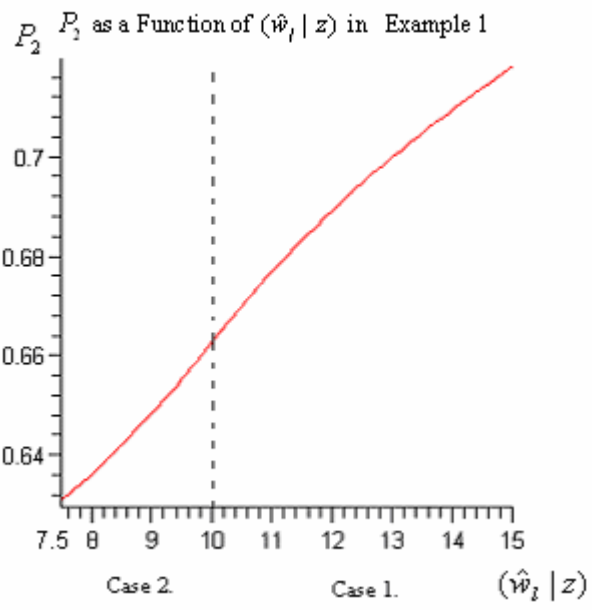
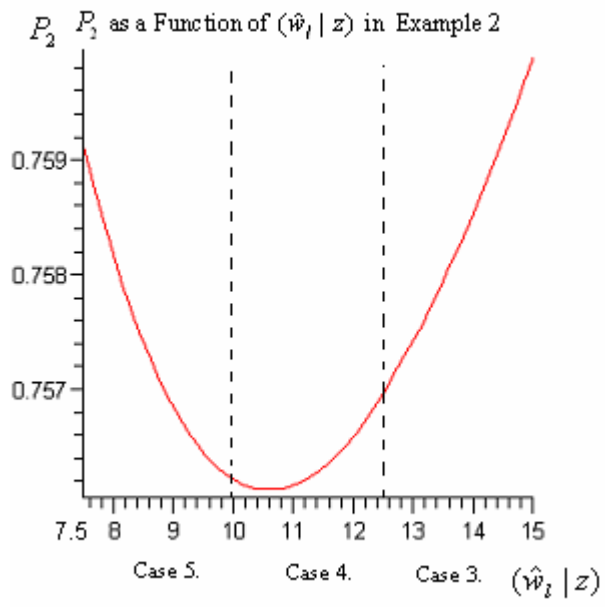


Figure 4.



### Appendix 1: Derivations of Results in Table 2

**A. Case 1:**  $0 < z < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{w_h + \hat{w}_l}{2}$ .

In this case,  $\hat{w}_l$  and  $\hat{w}_h$  are sufficiently high relative to  $z$  that only the households with both individuals unemployed are poor. The value of  $P_\alpha$  in this case is

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 \left(\frac{z - 0}{z}\right)^\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2. \quad (2)$$

Let us now see how  $P_\alpha$  is affected by an increase in  $\hat{w}_l$ . We have

$$\frac{dP_\alpha}{d\hat{w}_l} = 2(1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right) \left(-\frac{1}{2 - x_h}\right) \frac{dx_l}{d\hat{w}_l}. \quad (3)$$

For a standard labor demand function with  $\frac{dx_l}{d\hat{w}_l} < 0$ , (2) is always positive – that is, poverty

always increases as the minimum wage increases. If, furthermore, we assume a constant elasticity of labor demand  $\eta = -\frac{\hat{w}_l}{x_l} \frac{dx_l}{d\hat{w}_l} > 0$ , (2) can be manipulated to produce

$$\begin{aligned} \frac{\hat{w}_l}{x_l} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right) \left(-\frac{1}{2 - x_h}\right) \frac{dx_l}{d\hat{w}_l} \frac{\hat{w}_l}{x_l} \\ &= 2(1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right) \left(\frac{1}{2 - x_h}\right) \eta, \end{aligned}$$

in which it is apparent that  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  if and only if  $\eta > 0$  for all  $\alpha$ .

**B. Case 2:**  $0 \leq \frac{\hat{w}_l}{2} < z < \frac{w_h}{2} < \hat{w}_l < \frac{w_h + \hat{w}_l}{2} < w_h$ .

In Case 2, the poor households are those where both individuals are unemployed or where only one household member is employed and that person earns the minimum wage. In this case,

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha. \quad (4)$$

The effect of a higher minimum wage is obtained to be

$$\begin{aligned} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left[\left(1-\frac{x_l}{2-x_h}\right)\left(-1+\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right)-\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right]\frac{dx_l}{d\hat{w}_l} \\ &+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right). \end{aligned} \quad (5)$$

If in (5), we assume constant elasticity of labor demand as before, we have:

$$\begin{aligned} \frac{\hat{w}_l}{x_l} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left[\left(1-\frac{x_l}{2-x_h}\right)\left(-1+\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right)-\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right]\frac{\hat{w}_l}{x_l} \frac{dx_l}{d\hat{w}_l} \\ &+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right)\frac{\hat{w}_l}{x_l}, \end{aligned}$$

which can be manipulated to yield

$$\begin{aligned} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left[\left(1-\frac{x_l}{2-x_h}\right)\left(1-\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right)+\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right]\eta\left(\frac{\hat{w}_l}{x_l}\right)^{-1} \\ &+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right). \end{aligned} \quad (6)$$

The first term in (6) can be thought of as the unemployment effect; it tells us how an increase in the minimum wage brings about a reduction in employment. This term may be shown to be always positive as follows. The expression in brackets in the first term

$$\left[\left(1-\frac{x_l}{2-x_h}\right)\left(1-\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right)+\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right]$$

is always positive since  $0 \leq \left(1-\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right) \leq 1$  for all  $\alpha$ . This term is multiplied by a number of

positive terms, which proves that the entire first expression is always positive. The second term in

(6) can be thought of as the earnings effect; it tells us how an increase in the minimum wage

affects  $P_\alpha$  via the gain in earnings for those employed. To sign this expression, note that in Case

2,  $\frac{\hat{w}_l}{2} < z$ , hence  $\left(1-\frac{\hat{w}_l}{2z}\right) > 0$ , and therefore all terms are positive except for  $-\frac{1}{2z}$ . The

product of these terms is therefore negative.

To analyze the sign of  $\frac{dP_\alpha}{d\hat{w}_l}$ , let us deal now with some particular values of  $\alpha$ . First, it

may be shown that when  $\alpha = 0$ , for any  $\eta$ ,  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ . Equation (6) becomes

$$\frac{dP_0}{d\hat{w}_l} = 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left(\frac{x_l}{2-x_h}\right)\eta\left(\frac{\hat{w}_l}{x_l}\right)^{-1},$$

which is positive for any positive  $\eta$ . It may also be shown that when  $\alpha \geq 1$ ,  $\frac{dP_\alpha}{d\hat{w}_l} \geq (<)0$  if and

only if

$$\eta \geq (<) \frac{(1 - \frac{x_l}{2 - x_h})\alpha(1 - \frac{\hat{w}_l}{2z})^{\alpha-1}(\frac{1}{2z})\hat{w}_l}{[(1 - \frac{x_l}{2 - x_h}) + (\frac{2x_l}{2 - x_h} - 1)(1 - \frac{\hat{w}_l}{2z})^\alpha]}$$

**C. Case 3:**  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < z \leq \hat{w}_l < \frac{w_h + \hat{w}_l}{2}$ .

In this case, the poverty group consists of households in which both individuals are unemployed and those in which only one household member is employed regardless of the sector of employment. The extent of poverty in this case is given by

$$P_\alpha = (1 - x_h)(1 - \frac{x_l}{2 - x_h})^2 + 2(1 - x_h)(\frac{x_l}{2 - x_h})(1 - \frac{x_l}{2 - x_h})(1 - \frac{\hat{w}_l}{2z})^\alpha + x_h(1 - \frac{x_l}{2 - x_h})(1 - \frac{w_h}{2z})^\alpha. \quad (7)$$

Differentiating (7) with respect to the level of the minimum wage yields

$$\begin{aligned} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1 - x_h)(1 - \frac{x_l}{2 - x_h})(-\frac{1}{2 - x_h})\frac{dx_l}{d\hat{w}_l} + 2(1 - x_h)(\frac{1}{2 - x_h})\frac{dx_l}{d\hat{w}_l}(1 - \frac{x_l}{2 - x_h})(1 - \frac{\hat{w}_l}{2z})^\alpha \\ &+ 2(1 - x_h)(\frac{x_l}{2 - x_h})(-\frac{1}{2 - x_h})\frac{dx_l}{d\hat{w}_l}(1 - \frac{\hat{w}_l}{2z})^\alpha + 2(1 - x_h)(\frac{x_l}{2 - x_h})(1 - \frac{x_l}{2 - x_h})\alpha(1 - \frac{\hat{w}_l}{2z})^{\alpha-1}(-\frac{1}{2z}) \\ &+ (-\frac{1}{2 - x_h})x_h\frac{dx_l}{d\hat{w}_l}(1 - \frac{w_h}{2z})^\alpha. \end{aligned} \quad (8)$$

If the labor demand elasticity  $\eta$  is assumed to be constant, equation (8) can be further manipulated to yield a condition in terms of  $\eta$ :

$$\begin{aligned} \frac{\hat{w}_l}{x_l} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1 - x_h)(1 - \frac{x_l}{2 - x_h})(-\frac{1}{2 - x_h})\frac{\hat{w}_l}{x_l}\frac{dx_l}{d\hat{w}_l} + 2(1 - x_h)(\frac{1}{2 - x_h})\frac{\hat{w}_l}{x_l}\frac{dx_l}{d\hat{w}_l}(1 - \frac{x_l}{2 - x_h})(1 - \frac{\hat{w}_l}{2z})^\alpha \\ &+ 2(1 - x_h)(\frac{x_l}{2 - x_h})(-\frac{1}{2 - x_h})\frac{dx_l}{d\hat{w}_l}\frac{\hat{w}_l}{x_l}(1 - \frac{\hat{w}_l}{2z})^\alpha + 2(1 - x_h)(\frac{x_l}{2 - x_h})(1 - \frac{x_l}{2 - x_h})\alpha(1 - \frac{\hat{w}_l}{2z})^{\alpha-1}(-\frac{1}{2z})\frac{\hat{w}_l}{x_l} \\ &+ (-\frac{1}{2 - x_h})x_h\frac{dx_l}{d\hat{w}_l}(1 - \frac{w_h}{2z})^\alpha\frac{\hat{w}_l}{x_l}, \end{aligned}$$

which in turn produces



$$\begin{aligned}
\frac{dP_\alpha}{dw_l} &= \eta \left[ 2(1-x_h) \left(1 - \frac{x_l}{2-x_h}\right) \left(\frac{1}{2-x_h}\right) - 2(1-x_h) \left(\frac{1}{2-x_h}\right) \left(1 - \frac{x_l}{2-x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha \right. \\
&+ 2(1-x_h) \left(\frac{1}{2-x_h}\right) \left(\frac{x_l}{2-x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha + \left(\frac{1}{2-x_h}\right) x_h \left(1 - \frac{w_h}{2z}\right)^\alpha \left. \right] \left(\frac{\hat{w}_l}{x_l}\right)^{-1} \\
&+ 2(1-x_h) \left(\frac{x_l}{2-x_h}\right) \left(1 - \frac{x_l}{2-x_h}\right) \alpha \left(1 - \frac{\hat{w}_l}{2z}\right)^{\alpha-1} \left(-\frac{1}{2z}\right).
\end{aligned} \tag{9}$$

Again, the first term is the unemployment effect (which is always positive), and the second term is the earnings effect (which is always negative).

Let us look at particular values of  $\alpha$ . It may be verified that when  $\alpha = 0$ , for any  $\eta$ ,  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ . Furthermore, when  $\alpha \geq 1$ ,  $\frac{dP_\alpha}{d\hat{w}_l} \geq (<) 0$  if and only if

$$\eta \geq (<) \frac{[(1-x_h) \left(1 - \frac{x_l}{2-x_h}\right) \alpha \left(1 - \frac{\hat{w}_l}{2z}\right)^{\alpha-1} \left(\frac{\hat{w}_l}{z}\right)]}{\left[2(1-x_h) \left(1 - \frac{x_l}{2-x_h}\right) - 2(1-x_h) \left(1 - \frac{2x_l}{2-x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha + x_h \left(1 - \frac{w_h}{2z}\right)^\alpha\right]}.$$

**D. Case 4:**  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < z \leq \frac{w_h + \hat{w}_l}{2}$ .

In Case 4, households in which both individuals are unemployed and in which only one household member is employed are below the poverty line. Moreover, if both household members are employed and earn the minimum wage, that household falls below the poverty line. On the other hand, a household with a high wage earner and a low wage earner is above the poverty line. This could be a possible stylization of the US labor market where about 80% of minimum wage earners live with a high wage earner (Burkhauser, Couch, and Wittenburg, 2000).

The poverty measure in this case becomes:

$$\begin{aligned}
P_\alpha &= (1-x_h) \left(1 - \frac{x_l}{2-x_h}\right)^2 + 2(1-x_h) \left(\frac{x_l}{2-x_h}\right) \left(1 - \frac{x_l}{2-x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha + x_h \left(1 - \frac{x_l}{2-x_h}\right) \left(1 - \frac{w_h}{2z}\right)^\alpha \\
&+ (1-x_h) \left(\frac{x_l}{2-x_h}\right)^2 \left(1 - \frac{\hat{w}_l}{z}\right)^\alpha.
\end{aligned}$$

(10)

Differentiating (10) with respect to  $w_l$  to get the effect on  $P_\alpha$  of increase in  $w_l$ ,

$$\begin{aligned}
\frac{dP_\alpha}{dw_l} &= 2(1-x_h)\left(1-\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{dx_l}{d\hat{w}_l} + 2(1-x_h)\left(\frac{1}{2-x_h}\right)\frac{dx_l}{d\hat{w}_l}\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{dx_l}{d\hat{w}_l}\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right) \\
&+ \left(-\frac{1}{2-x_h}\right)x_h\frac{dx_l}{dw_l}\left(1-\frac{w_h}{2z}\right)^\alpha \\
&+ 2(1-x_h)\frac{x_l}{2-x_h}\left(\frac{1}{2-x_h}\right)\frac{dx_l}{dw_l}\left(1-\frac{\hat{w}_l}{z}\right)^\alpha + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\alpha\left(1-\frac{\hat{w}_l}{z}\right)^{\alpha-1}\left(-\frac{1}{z}\right)
\end{aligned} \tag{11}$$

If the labor demand elasticity  $\eta$  is assumed to be constant, equation (11) can be rewritten as:

$$\begin{aligned}
\frac{\hat{w}_l}{x_l}\frac{dP_\alpha}{dw_l} &= 2(1-x_h)\left(1-\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{\hat{w}_l}{x_l}\frac{dx_l}{d\hat{w}_l} + 2(1-x_h)\left(\frac{1}{2-x_h}\right)\frac{\hat{w}_l}{x_l}\frac{dx_l}{d\hat{w}_l}\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{dx_l}{d\hat{w}_l}\frac{\hat{w}_l}{x_l}\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right)\frac{\hat{w}_l}{x_l} \\
&+ \left(-\frac{1}{2-x_h}\right)x_h\frac{dx_l}{dw_l}\left(1-\frac{w_h}{2z}\right)^\alpha\frac{\hat{w}_l}{x_l} + 2(1-x_h)\frac{x_l}{2-x_h}\left(\frac{1}{2-x_h}\right)\frac{dx_l}{dw_l}\left(1-\frac{\hat{w}_l}{z}\right)^\alpha\frac{\hat{w}_l}{x_l} \\
&+ (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\alpha\left(1-\frac{\hat{w}_l}{z}\right)^{\alpha-1}\left(-\frac{1}{z}\right)\frac{\hat{w}_l}{x_l},
\end{aligned}$$

which can be expressed as:

$$\begin{aligned}
\frac{dP_\alpha}{dw_l} &= \eta\left[2(1-x_h)\left(1-\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right) - 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right. \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + \left.\left(\frac{1}{2-x_h}\right)x_h\left(1-\frac{w_h}{2z}\right)^\alpha - 2(1-x_h)\frac{x_l}{2-x_h}\left(\frac{1}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{z}\right)^\alpha\right]\left(\frac{\hat{w}_l}{x_l}\right)^{-1} \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right) \\
&+ (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\alpha\left(1-\frac{\hat{w}_l}{z}\right)^{\alpha-1}\left(-\frac{1}{z}\right).
\end{aligned} \tag{12}$$

Again, the first term on the right hand side is the unemployment effect. which can be shown to be always positive. (Group the first two terms in brackets together and the third and fifth terms together, from which we can see that the bracketed term is always positive.) The rest of the terms

of the equation form the earnings effect, which is always negative. Looking at different values of  $\alpha$ , when  $\alpha = 0$ , for any  $\eta$ ,  $\frac{dP_\alpha}{d\hat{w}_l} > 0$ . When  $\alpha \geq 1$ , it may be shown that  $\frac{dP_\alpha}{d\hat{w}_l} \geq (<) 0$  if and

only if

$$\eta \geq (<) \frac{[(1-x_h)(1-\frac{x_l}{2-x_h})\alpha(1-\frac{\hat{w}_l}{2z})^{\alpha-1}(\frac{\hat{w}_l}{z}) + (1-x_h)(\frac{x_l}{2-x_h})\alpha(1-\frac{\hat{w}_l}{z})^{\alpha-1}(\frac{\hat{w}_l}{z})]}{[2(1-x_h)(1-\frac{x_l}{2-x_h}) - 2(1-x_h)(1-\frac{2x_l}{2-x_h})(1-\frac{\hat{w}_l}{2z})^\alpha + x_h(1-\frac{w_h}{2z})^\alpha - 2(1-x_h)\frac{x_l}{2-x_h}(1-\frac{\hat{w}_l}{z})^\alpha]}$$

**E. Case 5:**  $0 < \frac{\hat{w}_l}{2} < \frac{w_h}{2} < \hat{w}_l < \frac{w_h + \hat{w}_l}{2} < z$ .

For Case 5, all households fall below the poverty line regardless of the employment status of the household members. The poverty measure can be expressed in this case as:

$$\begin{aligned} P_\alpha &= (1-x_h)(1-\frac{x_l}{2-x_h})^2 + 2(1-x_h)(\frac{x_l}{2-x_h})(1-\frac{x_l}{2-x_h})(1-\frac{\hat{w}_l}{2z})^\alpha + x_h(1-\frac{x_l}{2-x_h})(1-\frac{w_h}{2z})^\alpha \\ &+ (1-x_h)(\frac{x_l}{2-x_h})^2(1-\frac{\hat{w}_l}{z})^\alpha + x_h\frac{x_l}{2-x_h}(1-\frac{\hat{w}_l + w_h}{2z})^\alpha. \end{aligned} \quad (13)$$

Differentiating (13) with respect to  $\hat{w}_l$  yields

$$\begin{aligned} \frac{dP_\alpha}{d\hat{w}_l} &= 2(1-x_h)(1-\frac{x_l}{2-x_h})(-\frac{1}{2-x_h})\frac{dx_l}{d\hat{w}_l} + 2(1-x_h)(\frac{1}{2-x_h})\frac{dx_l}{d\hat{w}_l}(1-\frac{x_l}{2-x_h})(1-\frac{\hat{w}_l}{2z})^\alpha \\ &+ 2(1-x_h)(\frac{x_l}{2-x_h})(-\frac{1}{2-x_h})\frac{dx_l}{d\hat{w}_l}(1-\frac{\hat{w}_l}{2z})^\alpha + 2(1-x_h)(\frac{x_l}{2-x_h})(1-\frac{x_l}{2-x_h})\alpha(1-\frac{\hat{w}_l}{2z})^{\alpha-1}(-\frac{1}{2z}) \\ &+ (-\frac{1}{2-x_h})x_h\frac{dx_l}{d\hat{w}_l}(1-\frac{w_h}{2z})^\alpha \\ &+ 2(1-x_h)\frac{x_l}{2-x_h}(\frac{1}{2-x_h})\frac{dx_l}{d\hat{w}_l}(1-\frac{\hat{w}_l}{z})^\alpha + (1-x_h)(\frac{x_l}{2-x_h})^2\alpha(1-\frac{\hat{w}_l}{z})^{\alpha-1}(-\frac{1}{z}) \\ &+ x_h\frac{1}{2-x_h}\frac{dx_l}{d\hat{w}_l}(1-\frac{\hat{w}_l + w_h}{2z})^\alpha + x_h\frac{x_l}{2-x_h}\alpha(1-\frac{\hat{w}_l + w_h}{2z})^{\alpha-1}(-\frac{1}{2z}). \end{aligned} \quad (14)$$

If the elasticity of labor demand is assumed constant, (14) can be rewritten as:

$$\begin{aligned}
\frac{\hat{w}_l}{x_l} \frac{dP_\alpha}{dw_l} &= 2(1-x_h)\left(1-\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{\hat{w}_l}{x_l} \frac{dx_l}{d\hat{w}_l} + 2(1-x_h)\left(\frac{1}{2-x_h}\right)\frac{\hat{w}_l}{x_l} \frac{dx_l}{d\hat{w}_l} \left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(-\frac{1}{2-x_h}\right)\frac{dx_l}{d\hat{w}_l} \frac{\hat{w}_l}{x_l} \left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right)\frac{\hat{w}_l}{x_l} \\
&+ \left(-\frac{1}{2-x_h}\right)x_h \frac{dx_l}{dw_l} \left(1-\frac{w_h}{2z}\right)^\alpha \frac{\hat{w}_l}{x_l} + 2(1-x_h)\frac{x_l}{2-x_h}\left(\frac{1}{2-x_h}\right)\frac{dx_l}{dw_l} \left(1-\frac{\hat{w}_l}{z}\right)^\alpha \frac{\hat{w}_l}{x_l} \\
&+ (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2 \alpha\left(1-\frac{\hat{w}_l}{z}\right)^{\alpha-1}\left(-\frac{1}{z}\right)\frac{\hat{w}_l}{x_l} + x_h \frac{1}{2-x_h} \frac{dx_l}{dw_l} \frac{\hat{w}_l}{x_l} \left(1-\frac{\hat{w}_l+w_h}{2z}\right)^\alpha \\
&+ x_h \frac{x_l}{2-x_h} \alpha\left(1-\frac{\hat{w}_l+w_h}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right)\frac{\hat{w}_l}{x_l},
\end{aligned}$$

which in turn can be rewritten as

$$\begin{aligned}
\frac{dP_\alpha}{dw_l} &= \eta\left[2(1-x_h)\left(1-\frac{x_l}{2-x_h}\right)\left(\frac{1}{2-x_h}\right) - 2(1-x_h)\left(\frac{1}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha\right. \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(\frac{1}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha - 2(1-x_h)\frac{x_l}{2-x_h}\left(\frac{1}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{z}\right)^\alpha \\
&+ \left.\left(\frac{1}{2-x_h}\right)x_h\left(1-\frac{w_h}{2z}\right)^\alpha - x_h \frac{1}{2-x_h} \left(1-\frac{\hat{w}_l+w_h}{2z}\right)^\alpha \right]\left(\frac{\hat{w}_l}{x_l}\right)^{-1} \\
&+ 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\alpha\left(1-\frac{\hat{w}_l}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right) \\
&+ (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2 \alpha\left(1-\frac{\hat{w}_l}{z}\right)^{\alpha-1}\left(-\frac{1}{z}\right) + x_h \frac{x_l}{2-x_h} \alpha\left(1-\frac{\hat{w}_l+w_h}{2z}\right)^{\alpha-1}\left(-\frac{1}{2z}\right).
\end{aligned}$$

(15)

Again, we have the unemployment effect (always positive) in the first term of the right hand side of the equation and the earnings effect (always negative) in the rest of the equation.

Analyzing (15) for specific values of  $\alpha$ , when  $\alpha = 0$ , for any  $\eta$ ,  $\frac{dP_\alpha}{d\hat{w}_l} = 0$ . This is

because everyone is under the poverty line, and that does not change as  $\hat{w}_l$  increases.

When  $\alpha = 1$ , it is straightforward to show that for  $\eta \geq (<)1$ ,  $\frac{dP_\alpha}{d\hat{w}_l} \geq (<)0$ . Finally, for  $\alpha > 1$ , we

have the condition that:

$$\frac{dP_\alpha}{d\hat{w}_l} \geq (<)0 \text{ if and only if}$$

$$\begin{aligned}
& [(1-x_h)(1-\frac{x_l}{2-x_h})\alpha(1-\frac{\hat{w}_l}{2z})^{\alpha-1}(\frac{\hat{w}_l}{z}) + (1-x_h)(\frac{x_l}{2-x_h})\alpha(1-\frac{\hat{w}_l}{z})^{\alpha-1}(\frac{\hat{w}_l}{z}) \\
& + x_h\alpha(1-\frac{\hat{w}_l+w_h}{2z})^{\alpha-1}(\frac{\hat{w}_l}{2z})] \\
\eta \geq (<) & \frac{\phantom{[2(1-x_h)(1-\frac{x_l}{2-x_h}) - 2(1-x_h)(1-\frac{2x_l}{2-x_h})(1-\frac{\hat{w}_l}{2z})^\alpha + x_h(1-\frac{w_h}{2z})^\alpha - 2(1-x_h)\frac{x_l}{2-x_h}(1-\frac{\hat{w}_l}{z})^\alpha - x_h(1-\frac{\hat{w}_l+w_h}{2z})^\alpha]}}{[2(1-x_h)(1-\frac{x_l}{2-x_h}) - 2(1-x_h)(1-\frac{2x_l}{2-x_h})(1-\frac{\hat{w}_l}{2z})^\alpha + x_h(1-\frac{w_h}{2z})^\alpha - 2(1-x_h)\frac{x_l}{2-x_h}(1-\frac{\hat{w}_l}{z})^\alpha - x_h(1-\frac{\hat{w}_l+w_h}{2z})^\alpha]}.
\end{aligned}$$

## Appendix 2: Proofs of Propositions 1-3

### Proposition 1

Proof:

1.a) From (6),  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  within Case 2.

1.b) From (3),  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  within Case 1.

1.c) The boundary between Cases 2 and 1 occurs at  $\hat{w}_l = 2z$ . From (4),

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha \text{ in Case 2; from (2),}$$

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 \text{ in Case 1. Evaluated at } \hat{w}_l = 2z \text{ and setting } \alpha = 0,$$

$$P_0 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \quad (16)$$

in Case 2 and

$$P_0 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 \quad (17)$$

in Case 1. Because (16) > (17),  $P_0$  falls discontinuously at  $\hat{w}_l = 2z$ .

Combining results 1.a-c), Proposition 1 is proved. ||

### Proposition 2

Proof:

2.a) From (15),  $\frac{dP_\alpha}{d\hat{w}_l} = 0$  within Case 5.

2.b) From (12),  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  within Case 4.

2.c) From (9),  $\frac{dP_\alpha}{d\hat{w}_l} > 0$  within Case 3.

2.d) The boundary between Cases 5 and 4 occurs at  $\hat{w}_l = 2z - \hat{w}_h$ . From (13),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\left(1-\frac{\hat{w}_l}{z}\right)^\alpha + x_h\frac{x_l}{2-x_h}\left(1-\frac{\hat{w}_l+w_h}{2z}\right)^\alpha$$

in Case 5; from (10),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\left(1-\frac{\hat{w}_l}{z}\right)^\alpha$$

in Case 4. Evaluated at  $\hat{w}_l = 2z - \hat{w}_h$  and setting  $\alpha = 0$ ,

$$P_0 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right) + x_h\left(1-\frac{x_l}{2-x_h}\right) \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2 + x_h\frac{x_l}{2-x_h}$$

(18)

in Case 5 and

$$P_0 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right) + x_h\left(1-\frac{x_l}{2-x_h}\right) \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2$$

(19)

in Case 4. Because (18) > (19),  $P_0$  falls discontinuously at  $\hat{w}_l = 2z - \hat{w}_h$ .

2.e) The boundary between Cases 4 and 3 occurs at  $\hat{w}_l = z$ . From (10),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\left(1-\frac{\hat{w}_l}{z}\right)^\alpha$$

in Case 4; from (7),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha$$

in Case 3. Evaluated at  $\hat{w}_l = z$  and setting  $\alpha = 0$ ,

$$P_0 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right) + x_h\left(1-\frac{x_l}{2-x_h}\right) \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2$$

in Case 4 and

$$P_0 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right) + x_h\left(1-\frac{x_l}{2-x_h}\right) \quad (20)$$

in Case 3. Because (19) > (20),  $P_0$  falls discontinuously at  $\hat{w}_l = 2z - w_h$ .

Combining results 2.a-e), Proposition 2 is proved. |

### Proposition 3

Proof for  $z < \frac{w_h}{2}$ :

The continuity of  $P_2$  within each case is evident. As for the boundary, the dividing line between Cases 2 and 1 occurs at  $\hat{w}_l = 2z$ . From (4),

$$P_2 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha \text{ in Case 2; from (2),}$$

$$P_2 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 \text{ in Case 1. Evaluated at } \hat{w}_l = 2z, P_2 = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 \text{ in Case}$$

2, which is identical to what  $P_2$  equals in Case 1 at that point. Continuity is thereby proved. ||

Proof for  $z > \frac{w_h}{2}$ :

4.a-c) The continuity of  $P_2$  within each case follows exactly as in 2.a-c).

4.d) The boundary between Cases 5 and 4 occurs at  $\hat{w}_l = 2z - \hat{w}_h$ . From (13),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\left(1-\frac{\hat{w}_l}{z}\right)^\alpha + x_h\frac{x_l}{2-x_h}\left(1-\frac{\hat{w}_l + w_h}{2z}\right)^\alpha$$

in Case 5; from (10),

$$P_\alpha = (1-x_h)\left(1-\frac{x_l}{2-x_h}\right)^2 + 2(1-x_h)\left(\frac{x_l}{2-x_h}\right)\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{\hat{w}_l}{2z}\right)^\alpha + x_h\left(1-\frac{x_l}{2-x_h}\right)\left(1-\frac{w_h}{2z}\right)^\alpha \\ + (1-x_h)\left(\frac{x_l}{2-x_h}\right)^2\left(1-\frac{\hat{w}_l}{z}\right)^\alpha$$



in Case 4. Evaluated at  $\hat{w}_l = 2z - \hat{w}_h$  and setting  $\alpha = 2$ ,

$$P_2 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{2z - w_h}{2z}\right)^2 + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^2 \\ + (1 - x_h) \left(\frac{x_l}{2 - x_h}\right)^2 \left(1 - \frac{2z - w_h}{z}\right)^2$$

in Case 5 and

$$P_2 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{2z - w_h}{2z}\right)^2 + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^2 \\ + (1 - x_h) \left(\frac{x_l}{2 - x_h}\right)^2 \left(1 - \frac{2z - w_h}{z}\right)^2$$

in Case 4. These are identical, and therefore  $P_2$  is continuous at the boundary between Cases 5 and 4.

4.e) The boundary between Cases 4 and 3 occurs at  $\hat{w}_l = z$ . From (10),

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^\alpha \\ + (1 - x_h) \left(\frac{x_l}{2 - x_h}\right)^2 \left(1 - \frac{\hat{w}_l}{z}\right)^\alpha$$

in Case 4; from (7),

$$P_\alpha = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{\hat{w}_l}{2z}\right)^\alpha + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^\alpha$$

in Case 3. Evaluated at  $\hat{w}_l = z$  and setting  $\alpha = 2$ ,

$$P_2 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(\frac{1}{2}\right)^2 + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^2 \text{ in}$$

Case 4 and

$$P_2 = (1 - x_h) \left(1 - \frac{x_l}{2 - x_h}\right)^2 + 2(1 - x_h) \left(\frac{x_l}{2 - x_h}\right) \left(1 - \frac{x_l}{2 - x_h}\right) \left(\frac{1}{2}\right)^2 + x_h \left(1 - \frac{x_l}{2 - x_h}\right) \left(1 - \frac{w_h}{2z}\right)^2 \text{ in}$$

Case 3. These are identical, and therefore  $P_2$  is continuous at the boundary between Cases 4 and 3.

Combining results 4.a-e), Proposition 4 is proved. ||